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Statistics of Colored Flux Lines

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A d-dimensional model of fluctuating flux lines with hard-core interaction is considered. For $d=3$ this is a description of the Abrikosov flux phase in superconducting systems with short coherence length. Introducing flux lines with different colors, one can solve the limit of infinitely many colors. This solution, which describes free fermions, is a mean-field approximation in the case of a finite number of colors. For $d \leqslant 3$ this solution agrees with the result of a perturbative treatment of hard-core bosons. Moreover, the entanglement of the flux lines has only an irrelevant effect of the colored flux lines.

KEY WORDS: Second-order phase transition; vortex phase; fermion–boson transmutation.

1. INTRODUCTION

The statistics of flux lines (FLs) is related to a number of classical problems in statistical physics ranging from the statistics of dimers[1] on a d-dimensional lattice to directed walks or directed polymers[2] with excluded-volume interaction. Recently it has attracted attention in solid-state physics due to its realization as the Abrikosov flux phase[3] of high-T_c superconductors. Nelson[4] considered the statistics of FLs as a problem of interacting Bose world lines. The analogy of FLs and Bose World lines is obvious from the path integral representation of both systems. FLs, subject to thermal fluctuations, behave like directed polymers. There is a path integral for the latter where the internal parametrization by the lengths of the polymer or FL appears as the imaginary time of the thermal Bose system. Perturbation theory around noninteracting Bose world lines (random walks) is rather complicated. In particular, a sensible result can

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only be obtained from Gaussian fluctuations around a trivial mean-field result.[4]

We started in a previous article from a different point[5]: Considering hard-core FLs, one can construct an algebra (η algebra) which describes the statistics of a grand canonical system. The idea is similar to quantum statistics, where one can describe fermions by a Grassmann algebra.[6] In contrast to a Grassmann algebra for free fermions or the Gaussian field theory of a complex field in the case of free bosons, the field theory on the η algebra cannot be solved exactly. However, from the η algebra it becomes obvious that one can write the interacting FLs in terms of fermions subject to a random field.[5] In the present article we will investigate a saddle-point analysis and the corresponding perturbation theory of this model. For this purpose, we introduce fermions with N different colors. This corresponds to a system of FLs with $N(2N-1)$ colors. The model can be solved in the limit $N \to \infty$. The solution is a grand canonical system of free fermions. For finite N, we may apply a $1/N$ expansion. The Gaussian fluctuations around the mean-field solution $[O(1/N)]$ are massive even at the critical point, where the density of FLs vanishes. In contrast, the fluctuations around the free bosons in Nelson's case are a relevant perturbation with a vanishing mass at the critical point. Moreover, the interaction among these fluctuations is irrelevant only for $d > 3$. The advantage of the fermion description of the hard-core FLs is not surprising: in $d = 2$ it was well known[7] that free-fermion world lines are equivalent to hard-core FLs. In higher dimensions the hard-core interaction is still treated properly by the free-fermion saddle point. On the other hand, the free fermions do not describe the entanglement of FLs properly, since they carry a phase due to their anticommuting property. This is corrected by the fluctuations of the random field coupled to the fermions. Fortunately, it turns out that these fluctuations (and, therefore, the topological effect of the entanglement) are an irrelevant perturbation of the free fermions if N is large or for any N if $d \leqslant 3$. Thus, the free-fermion saddle point is a better approximation than the free-boson saddle point if $d \leqslant 3$.

A typical configuration of FLs is shown in Fig. 1. The density of FLs

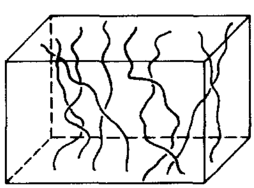

Fig. 1. A typical configuration of flux lines.

is controlled by a fugacity $\bar{\mu}$ which depends on physical parameters of the model. For a type II superconductor in a magnetic field H, for instance, this fugacity is proportional to $\exp[-\beta\lambda\Phi_0(H_{c1}-H)/4\pi)]$, where β is the inverse temperature, λ the magnetic penetration length of the superconductor, and Φ_0 the flux quantum.[3] The lower critical field H_{c1} indicates the transition from the Abrikosov phase $(H > H_{c1})$, where we have a nonzero density of FLs, to the Meissner phase $(N < H_{c1})$. The order parameter which characterizes both phases is the density of FLs, $n(\bar{\mu})$. There are no order-parameter fluctuations in the Meissner phase, because it costs too much energy to create even a single FL. Such an asymmetric phase transition is known also from dimers on a brick lattice,[1] where the excitations from the ground state are chains of turned dimers.

We know from a general random walk argument by Fisher[1] that the density of FLs obeys a power law $n(\bar{\mu}) \sim n_0(\bar{\mu}_c - \bar{\mu})^{\bar{\beta}}$ with

$$\bar{\beta} = (d-1)/2 \qquad \text{if} \quad d \leqslant 3 \tag{1.1}$$

This result will be recovered from the saddle-point calculation in this article. Furthermore, we will calculate the density–density correlations, which are characterized by a correlation length $\xi_{||}$ for correlations parallel to the direction of the FLs and $\xi_{\perp}$ for correlations perpendicular to the direction of the FLs. These correlation lengths obey also a power law with exponents

$$\nu_{||} = -1 \tag{1.2a}$$

and

$$\nu_{\perp} = -1/2 \tag{1.2b}$$

As an example for a system of FLs, the London limit of a type II superconductor in the presence of thermal fluctuations is considered in Section 2. Then we define our model of interacting FLs on a lattice in Section 2.1. The special case of noninteracting FLs is briefly discussed in Section 2.2. In Section 3 we introduce an algebra (η algebra) which describes correctly the statistics of hard-core FLs and show in Section 4 that this representation is related to the statistics of fermions coupled to a random field. In order to obtain a soluble limit of the system of FLs, we generalize our model by the introduction of colored FLs in Section 5. It turns out that the limit of infinitely many colors can be solved (Section 6). Finally, we show in Section 7 that the result of this limit is not disturbed in a relevant manner if we consider only a finite number of colors; i.e., it should be valid even for the original model with only one sort (color) of FLs.

2. FLUX LINES IN A SUPERCONDUCTOR

A type II superconductor in a sufficiently strong magnetic field is characterized by the superconducting order parameter and by the magnetization due to flux penetration. The description of this system simplifies essentially for an extreme type II superconductor, where the magnetic penetration length is very large compared to the coherence length of the superconducting state. In this case we may apply the "London model," in which one ignores the fluctuations of the superconducting order parameter[3] (i.e., the order-parameter field is homogeneous). This might be a good starting point for a phenomenological theory of the Abrikosov (or flux) phase in the high-T_c superconductors, since we do not know what the theory for the superconducting order parameter (e.g., Ginsburg–Landau theory) of these materials is. On the other hand, it is well known that most of the high-T_c superconductors are extreme type II materials.

The London model describes a system of n interacting vortices (or FLs) in an external magnetic field which is applied in the z direction. The jth FL is defined by its coordinates $r_j(z)$ perpendicular to the external magnetic field along the z direction. At very low temperatures the coordinates $\{r_j(z)\}$ do not vary with z; i.e., the FLs form an Abrikosov flux lattice. However, for higher temperatures the FLs fluctuate if the thermal energy is of the order of the stiffness of the FL. Following the literature, the Gibbs free energy of n interacting FLs is given as[3]

$$G_n = \sum_{i=1}^{n} \varepsilon_1 \int_0^L \left[1 + \left| \frac{dr_i(z)}{dz} \right|^2 \right]^{1/2} dz + \frac{g}{2} \sum_{i \neq j} \int_0^L K_0 \left[\frac{|r_i(z) - r_j(z)|}{\lambda} \right] dz - \frac{H}{4\pi} \int b(r, z)\, d^2r\, dz \tag{2.1}$$

The first term is the energy of a single FL, the second term is the energy of the interaction of pairs of FLs, and the third terms is the energy of the interaction of the FLs, which produce the internal magnetic field b in the superconductor, with the external magnetic field H. Here ε_1 is the energy per unit length of a FL, λ is the magnetic penetration length, and the coupling constant g is related to the flux quantum $\Phi_0 = 2\pi\hbar c/2e$ and λ by

$$g = \frac{\Phi_0^2}{8\pi^2\lambda^2} \tag{2.2}$$

The function K_0 is the modified Bessel function,[8] which is exponentially decaying for large arguments: $K_0(x) \sim (\pi/2x)^{1/2} e^{-x}$ for $x \sim \infty$. Now we may apply two approximations in order to simplify the expression of G_n:

(a) Local fluctuations of the FLs are weak:

$$\left[1+\left|\frac{dr_i(z)}{dz}\right|^2\right]^{1/2} \approx 1+\frac{1}{2}\left|\frac{dr_i(z)}{dz}\right|^2 \tag{2.3}$$

(b) Approximation of the short-range interaction by a hard core on a lattice Λ:

$$K_0\left[\frac{|r_i(z)-r_j(z)|}{\lambda}\right] \approx V_\lambda\left[\frac{|r_i(z)-r_j(z)|}{\lambda}\right] \tag{2.4}$$

with

$$(r_j, z) \in \Lambda \tag{2.5}$$

where the components are $(r_j)_i = 1, 2,..., N$, $z = 1, 2,..., L$, and

$$V_\lambda(x) = \begin{cases} \infty & \text{for} \quad x = 0 \\ 0 & \text{for} \quad x \neq 0 \end{cases} \tag{2.6}$$

The Gibbs free energy then reads

$$G_n = \left(\varepsilon_1 - \frac{H\Phi_0}{4\pi}\right) nL\lambda + \frac{\varepsilon_1}{2\lambda} \sum_{z=1}^{L} \sum_{i=1}^{n} |r_i(z+1) - r_i(z)|^2 + \frac{g\lambda^2}{2} \sum_{i \neq j} \sum_{z=1}^{L} V_\lambda\left[\frac{|r_i(z)-r_j(z)|}{\lambda}\right] \tag{2.7}$$

All lengths are measured in units of the penetration length λ. To describe the general situation under the influence of thermal fluctuations, we introduce a grand canonical ensemble of FLs. This is defined at the inverse temperature β by the partition function

$$Z = \sum_{n \geq 0} \frac{1}{n!} \sum_{\{r_j(z)\}_n} \exp(-\beta G_n) \tag{2.8}$$

The summation goes over all possible configurations $\{r_j(z)\}_n$ of FLs. Apparently, this model may undergo a phase transition when the "chemical potential" $\varepsilon_1 - H\Phi_0/4\pi$ of the fugacity $\zeta = \exp[-\beta\lambda(\varepsilon_1 - H\Phi_0/4\pi)]$ changes its sign. This phase transition is trivial in the sense that it is driven only by the fugacity. Defining the lower critical magnetic field H_{c1} by

$$H_{c1} = 4\pi\varepsilon_1/\Phi_0 \tag{2.9}$$

the system is in the Meissner phase (no FLs) if $H < H_{c1}$ (i.e., $\zeta < 1$) and in the Abrikosov phase (nonvanishing density of FLs) if $H > H_{c1}$ (i.e., $\zeta > 1$).

The transition from the Meissner to the Abrikosov phase has special properties which are different from most other phase transitions in statistical physics. It is related to the fact that the Meissner phase is empty because it needs too much energy to create a FL. Therefore, there are no fluctuations in this phase. On the other hand, thermodynamic quantities (e.g., specific heat) are divergent if one approaches the Meissner phase from the Abrikosov phase. Such an asymmetric behavior is also known from dimer models on special lattices[1] (e.g., brick lattice). Due to the lattice structure, it is only possible to change the direction of the dimers along a line through the entire lattice. This corresponds with the creation of a FL.

We will discuss in the following mainly the dilute region of the FL system. That means we are restricted to the neighborhood of the Abrikosov–Meissner transition and to those properties of the FL statistics which are related to long-range fluctuations. The weight of a FL element is given as

$$\exp\left[-\beta\frac{\varepsilon_1}{2\lambda}|r_i(z+1)-r_i(z)|^2\right] \tag{2.10}$$

At low temperatures the weight of line elements parallel to the z direction $[r_j(z+1)=r_j(z)]$ is dominant, whereas for higher temperatures also other line elements are important. In particular, near the phase transition the system of FLs is dilute and strongly fluctuating. Therefore, we may restrict our model to line elements which connect only nearest neighbor points r', r and ignore line elements which are parallel to the z axis. The weight of a FL element $((r, z), (r', z+1))$ reads then

$$\tau_{r,z;r',z+1}=\exp\left(-\beta\frac{\varepsilon_1}{2\lambda}|r'-r|^2\right)=\tau(\beta,\varepsilon_1)\sum_{j=1}^{2(d-1)}\delta_{r',r+e_j} \tag{2.11}$$

with

$$\tau(\beta,\varepsilon_1)=\exp\left(-\beta\frac{\varepsilon_1\lambda}{2}\right) \tag{2.12}$$

e_j is the lattice unit vector in j direction perpendicular to the z axis. The restriction to nearest neighbor FL elements is not crucial when we consider effects on length scales which are large compared to the lattice spacing λ, since all types of fluctuations can be presented by only nearest neighbor line elements. The advantage of this simplified description is the fact that there are only two quantities which depend on the physical parameters, namely τ and ζ. The partition function reads

$$Z=\sum_{n\geq 0}\frac{1}{n!}(\zeta\tau)^{nL}\sum_{\{r_j\}_n}\exp\left\{-\beta\frac{g\lambda^2}{2}\sum_{i\neq j}\sum_{z=1}^{L}V_\lambda\left[\frac{|r_i(z)-r_j(z)|}{\lambda}\right]\right\} \tag{2.13}$$

We notice that the interaction term does not depend on the parameters, due to the hard-core potential V_λ. The product $\zeta\tau$ appears as an effective weight of the FL element. Defining the new parameter

$$\bar{\mu} = [2(d-1)\,\zeta\tau]^{-1} \tag{2.14}$$

and neglecting the trivial factor

$$\bar{\mu}^{N^{d-1}L}$$

which does not depend on the configurations, we can write the partition function on the lattice as

$$\begin{aligned} Z &= \sum_{n \geq 0} \frac{1}{n!} [2(d-1)]^{-nL} \bar{\mu}^{(N^{d-1}-n)L} \\ &= \sum_{\{r_j\}_n} \exp\left\{ -\beta \frac{g\lambda^2}{2} \sum_{i \neq j} \sum_{z=1}^{L} V_\lambda \left[\frac{|r_i(e) - r_j(z)|}{\lambda} \right] \right\} \end{aligned} \tag{2.15}$$

Here we have separated the weight of a FL element $[2(d-1)^{-1}]$ and the weight $(\bar{\mu})$ of empty sites. Thus, we have obtained a simple description of the flux (or Abrikosov) phase which will be used for the further investigations.

2.1. A Model for Fluctuating Flux Lines

As discussed in the previous section, it is convenient to describe the statistics of FLs on a lattice. Then we avoid certain difficulties related to the regularization of a continuum model from the beginning. The model, which we have obtained from the London theory of type II superconductors after some approximation, has other realizations in statistical physics. Therefore, we summarize its definition here.

Since we are interested in the description of FLs with short-range interaction, the lattice spacing corresponds to the characteristic interaction length of the physical system. In a type II superconductor, for instance, this would be the magnetic penetration length, at least for directions perpendicular to the external magnetic field. For simplicity we will also use this lattice constant in the direction parallel to the magnetic field. These are physical reasons for the introduction of a lattice model. On the other hand, qualitative properties of the statistical model may not be sensitive to structures on short scales. This is in particular the case when we consider the transition from the Abrikosov to the Meissner phase, where long-range fluctuations dominate the behavior of the FLs near the transition point. Thus, the lattice spacing is irrelevant for the discussion of critical (asymptotic) properties. To be more specific, we define a d-dimensional cubic lattice as

$$\Lambda = \{1,..., N\} \times \{1,..., N\} \times \cdots \times \{1,..., N\} \times \{1,..., L\} \tag{2.16}$$

where the z direction is along $1, 2,..., L$. The vector $r = (x_1, x_2,..., x_{d-1})$ is perpendicular with components $x_j \in \{1, 2,..., N\}$. The simplest statistics of a FL $\{r(z)\}$ is now defined on Λ by its FL elements which connect lattice points (r, z) with $(r + e_j, z + 1)$. The probability of such a FL element is the same in any direction of the lattice unit vector e_j:

$$\text{Prob}((r, z) \to (r + e_j, z + 1)) = 1/2(d - 1) \tag{2.17}$$

FLs constructed with these elements have a fixed length and are flexible. A generalization to FLs with variable length will be given elsewhere. However, we do not expect a qualitative change of the properties on large scales.

A FL always starts at the bottom $z = 1$ and terminates at $z = L$. Then we introduce the hard-core interaction between the FLs: FLs do not touch or intersect each other; i.e., different FLs may not occupy the same lattice site (r, z). It is desirable to construct the FL statistics in a way such that the continuum limit leads also to a hard-core interaction. Therefore, we must avoid that FLs can cross each other between the lattice points. This can happen when two lines occupy nearest neighbor points for a given z. Lets assume these points are $(x_1,..., x_j,..., x_{d-1}, z)$ and $(x_1,..., x_j + 1,..., x_{d-1}, z)$. Then a crossing occurs between z and $z + 1$ when a FL continues from z to $z + 1$ as

$$(x_1,..., x_j,..., x_{d-1}, z) \to (x_1,..., x_j + 1,..., x_{d-1}, z + 1) \tag{2.18a}$$

and another one as

$$(x_1,..., x_j + 1,..., x_{d-1}, z) \to (x_1,..., x_j,..., x_{d-1}, z + 1) \tag{2.18b}$$

(see also Fig. 2). Such a situation, however, can be circumvented when we impose appropriate boundary conditions at $z = 1$: FLs can only start at

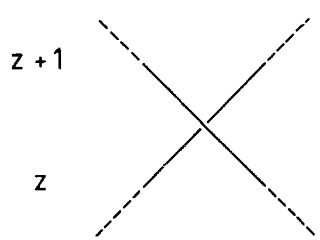

Fig. 2. Crossing between lattice points. This situation can be avoided by appropriate boundary conditions (see Section 2.1).

$z=1$ from points with odd coordinates x_j. Due to our choice of the FL elements in (2.13), this means that for z even the x_j coordinates are even and vice versa. Then FLs cannot sit on nearest neighbor sites and the crossing between lattice points is avoided.

The FLs should not cover the whole lattice. Therefore, we introduce a fugacity $\bar{\mu}$ which measures the weight of empty sites on the lattice. Thus, we have defined a grand canonical system of FLs on the lattice Λ.

Before we investigate the general model of FLs as defined above, we should mention that there are three special cases which are exactly soluble. The first case is the limit of noninteracting FLs. Then there is the case of hard rods where we ignore the thermal fluctuations of the FLs. And finally, we have the two-dimensional system; i.e., FLs on a square lattice. The last case is relevant as a model for domain boundaries on a surface. This model is soluble because it is equivalent to free fermions on the square lattice.[7] The latter is based on the fact that the FLs are topologically simple because they cannot wind around each other. However, they are subject to thermal fluctuations and interaction, in contrast to the first two examples. Therefore, the last case might be closest to the general model of FLs in d dimensions; in particular, to the superconductor, where $d=3$.

2.2. Noninteracting Flux Lines

As a simple example, we discuss the partition function (2.15) in the case of noninteracting (independent) FLs. A collapse of this grand canonical system can be avoided by introducing the constraint that at most one line per site can start at $z=1$, which corresponds to a surface interaction. The evaluation of Z is then only a simple combinatoric problem. Nevertheless, this case is interesting for our investigation of interacting FLs, since we can study the effect of the interaction by comparing with the results of the noninteracting system. We will show now that the noninteracting FLs exhibit a discontinuous behavior of the density of FLs (a step as a function of the fugacity $\bar{\mu}$) for the Abrikosov–Meissner transition at $\bar{\mu}=1$.

According to the model, a FL starts at $z=1$ and terminates at $z=L$. Suppose a flux line starts at $(r, 1)$. There are $2(d-1)$ ways to proceed to $z=2$. In general, there are always $2(d-1)$ ways to go from z to $z+1$. This implies $[2(d-1)]^L$ possible realizations of FLs starting from one site $(r, 1)$. If we have n FLs starting from fixed different sites at $z=1$, there are $[2(d-1)]^{Ln}$ possible realizations because there is no interaction among the FLs in the bulk $z>1$. Finally, we can choose

$$\binom{N^{d-1}}{n} \tag{2.19}$$

possible sites as starting points for FLs at $z=1$. Notice that the $n!$ of the partition function Z is here already incorporated. Therefore, we get

$$\binom{N^{d-1}}{n}[2(d-1)]^{Ln} \tag{2.20}$$

realizations of n FLs on the lattice. Then the partition function reads

$$Z=\sum_{n=0}^{N^{d-1}}\binom{N^{d-1}}{n}\bar{\mu}^{L(N^{d-1}-n)}=[1+\bar{\mu}^{L}]^{N^{d-1}} \tag{2.21}$$

Since $\bar{\mu}$ is the weight of empty sites, one obtains from the free energy $F=(1/N^{d-1}L)\log Z$ the density of noninteracting FLs as

$$n(\bar{\mu})=1-\bar{\mu}\frac{\partial F}{\partial \bar{\mu}}=1-(1+\bar{\mu}^{-L})^{-1}=(1+\bar{\mu}^{L})^{-1}\sim\begin{cases}0 & \text{if} \quad \bar{\mu}>1\\ 1 & \text{if} \quad \bar{\mu}<1\end{cases} \tag{2.22}$$

for $L\sim\infty$.

3. THE η ALGEBRA

An important fact in quantum statistics is that the field of a free particle is characterized by a special algebra according to the nature of the particle. For instance, bosons are given by a complex field, while fermions are decribed by a Grassmann field. Free fermions are hard-core particles, since more than one fermion cannot occupy the same site. This is represented by the Grassmann algebra. With respect to the interaction, hard-core FLs behave like free fermions. On the other hand, fermions are anticommutative objects, whereas FLs are commutative, like bosons. Thus, FLs can be considered as world lines of particles which have bosonic as well as fermionic properties. We will construct in the following an algebra which describes the FLs correctly.

On the lattice Λ, we introduce an algebra of variables $\{\eta_{r,x}, \bar{\eta}_{r,z}\}$ over the complex numbers. This algebra is chosen to represent the properties of the hard-core FL model defined in Section 2. The following properties are sufficient:

(i) The multiplication of $\eta_{r,z}, \bar{\eta}_{r,z}$ is commutative.

(ii) $\eta_{r,z}, \bar{\eta}_{r,z}$ are nilpotent [i.e., $(\eta_{r,z})^l=(\bar{\eta}_{r,z})^l=0$ for $l>1$].

(iii) There is a linear mapping (which we will call integration) $\int$ into the complex numbers with

$$\int\prod_{(r,z)\in\Lambda'\subseteq\Lambda}\eta_{r,z}\bar{\eta}_{r,z}=\begin{cases}1 & \text{if} \quad \Lambda'=\Lambda\\ 0 & \text{if} \quad \Lambda'\neq\Lambda\end{cases} \tag{3.1}$$

The result of the integration, the integral, is nonzero only if the product of the variables $\{\eta_{r,z}, \bar{\eta}_{r,z}\}$ on A is complete. The construction of analytic functions (e.g., the exponential function) is obvious in terms of polynomials of the $\eta_{r,z}$.

If is obvious that these variables are closely related to Grassmann variables. Writing for the latter $\{\psi_{1,r,z}, \bar{\psi}_{1,r,z}, \psi_{2,r,z}, \bar{\psi}_{2,r,z}\}$, the former variables can be expressed as products of the Grassmann variables:

$$\eta_{r,z} = \psi_{1,r,z}\psi_{2,r,z} \tag{3.2a}$$

$$\bar{\eta}_{r,z} = \bar{\psi}_{1,r,z}\bar{\psi}_{2,r,z} \tag{3.2b}$$

By means of the variables $\{\eta_{r,z}, \bar{\eta}_{r,z}\}$we define the following "weights":

I. Each FL element is identified with the "weight"

$$w_{r,z;r',z'}\eta_{r,z}\bar{\eta}_{r',z'} \tag{3.3}$$

where the matrix elements of w are given as

$$w_{r,z;r',z'} = \begin{cases} 1/2(d-1) & \text{for} \quad r' = r \pm e_j, \quad z' = z+1 \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

II. Furthermore, we introduce a local "weight" which corresponds to the fugacity $\bar{\mu}_{r,z}$ of the empty lattice sites:

$$\bar{\mu}_{r,z}\eta_{r,z}\bar{\eta}_{r,z} \tag{3.5}$$

By mean of I and II we can write the "weight" of an arbitrary configuration of FLs $I = \{r_i(z)\}$ as

$$W_I = \prod_{z=1}^{L} \prod_{r} (1 + \bar{\mu}_{r,z}\eta_{r,z}\bar{\eta}_{r,z}) \prod_{r_i(z) \in I} w_{r_i(z),z;r_i(z+1),z+1}\eta_{r_i(z),z}\bar{\eta}_{r_i(z+1),z+1} \tag{3.6}$$

It is convenient to impose periodic boundary conditions in the z direction and free boundary conditions in the other directions. These boundary conditions imply that we sum endpoints ($z = L$) of the FLs over permutations of the starting points ($z = 1$). Free boundary conditions in the z direction, however, can be obtained by introducing additional "weights" on the surface $z = 1$, L in order to get complete products according to (3.1). Since we are interested in the thermodynamic limit, the discussion of the boundary conditions is not so important.

We obtain the statistics of FLs from the integral over the algebra: the statistical weight P_I of a configuration I is

$$P_I = \int W_I \Big/ \sum_{\{I\}} \int W_I =: \frac{1}{Z} \int W_I \tag{3.7}$$

where $\sum_{\{I\}}$ is the summation over all possible configurations on Λ. Property (ii) implies that lines do not touch or intersect each other, while property (iii) guarantees that they must start and terminate on the surface $z=1$, L. Finally, the weights are always positive due to property (i). The partition function Z then reads

$$Z=\sum_{\{I\}}\int W_I=\int\prod_{z=1}^{L}\prod_r\left[(1+\bar{\mu}_{r,z}\eta_{r,z}\bar{\eta}_{r,z})\prod_{r'}(1+w_{r,z;r',z+1}\eta_{r,z}\bar{\eta}_{r',z+1})\right] \tag{3.8}$$

With (i)–(iii) we obtain then

$$=\sum_\pi\prod_{z,r}(w_{r,z;\pi(r,z)}+\bar{\mu}_{r,z}\delta_{r,z;\pi(r,z)}) \tag{3.9}$$

π are here the permutations of the lattice sites (r, z). Thus, the partition function of the hard-core FLs is a permanent. The rhs of (3.8) can also be written in the standard form of a partition function of statistical mechanics as

$$Z=\int\exp\left\{\sum_{r,r',z,z'}w_{r,z;r',z'}\eta_{r,z}\bar{\eta}_{r',z'}+\sum_{r,z}\bar{\mu}_{r,z}\eta_{r,z}\bar{\eta}_{r,z}\right\}=:\int\exp(-S) \tag{3.10}$$

The fugacity $\bar{\mu}_{r,z}$ appears here as a chemical potential of the grand canonical ensemble. Now we may also express expectation values of the statistical ensemble in terms of the variables η. For instance, the density of FLs per lattice site reads

$$n_{r,z}=\frac{1}{Z}\sum_{\{I\}}\int(1-\bar{\mu}_{r,z}\eta_{r,z}\bar{\eta}_{r,z})W_I=1-\frac{\bar{\mu}_{r,z}}{Z}\int\eta_{r,z}\bar{\eta}_{r,z}\exp(-S) \tag{3.11}$$

since we have, due to (ii) and (iii),

$$\frac{1}{Z}\int(1-\bar{\mu}_{r,z}\eta_{r,z}\bar{\eta}_{r,z})W_I=\begin{cases}P_I & \text{if} \quad (r,z)\in I\\ 0 & \text{if} \quad (r,z)\notin I\end{cases} \tag{3.12}$$

For the density–density correlation function we find accordingly

$$C_{r,z;r',z'}=\frac{1}{Z}\int(1-\bar{\mu}_{r,z}\eta_{r,z}\bar{\eta}_{r,z})(1-\bar{\mu}_{r',z'}\eta_{r',z'}\bar{\eta}_{r',z'})\exp(-S)-n_{r,r}n_{r',z'} \tag{3.13}$$

Unfortunately, we cannot evaluate the expression in (3.9), (3.11), or (3.13) exactly for any dimension d. Only for $d=2$ can the partition function (3.9) be determined due to its analogy to the free fermions discussed in the previous section. Then the permanent of (3.9) is identical to the fermion determinant:

$$\sum_{\pi} \prod_{z,r} (w+\bar{\mu})_{r,z;\pi(r,z)} = \sum_{\pi} (-1)^{\pi} \prod_{z,r} (w-\bar{\mu})_{r,z;\pi(r,z)} = \det(w-\bar{\mu}) \quad (3.14)$$

This is based on the fact that one can use the Grassmann representation of the η, $\bar{\eta}$ given in (3.2a), (3.2b). For $d=2$ we may integrate then over ψ_2 and $\bar{\psi}_2$. In higher dimensions, however, there appears a minus sign, depending on the special configuration, since we must order the product of Grassmann variables along the direction of the FLs. Due to the anticommutative relations among these variables, this leads for $d>2$ to a minus sign for certain configurations. In other words, FLs are topologically different from free-fermion world lines. The latter change sign if they exchange positions, whereas the former do not have a phase dependence. Therefore, FLs behave like boson world lines. This observation has led to the idea of treating the FL problem as a model of free bosons subject to a perturbation theory for the interaction. This means that we regard the distinction of different topological properties as important and treat the interaction as a perturbation. It turned out from perturbation theory that the latter is irrelevant only above three dimensions.[4] The representation of FLs which interact via a hard-core potential by the η algebra enables us to introduce a fermion representation.[5] The details and the properties of this representation will be discussed in the rest of this article.

4. FERMION REPRESENTATION OF FLUX LINES

In Section 3 we have seen that the partition function Z of a grand canonical system of hard-core FLs is given by a permanent:

$$Z = \sum_{\pi} \prod_{x} (w+\bar{\mu})_{x,\pi(x)}, \qquad x=(r,z) \quad (4.1)$$

In general, we may introduce a local fugacity $\bar{\mu}_x$. Local expectation values, e.g., the local average density of FLs or the density–density correlation function, can then be obtained by differentiation of $\log Z$ with respect to the local fugacity $\bar{\mu}_x$. For instance, we get with (3.11) for the density of FLs

$$n_x = 1 - \bar{\mu}_x \frac{\partial}{\partial \bar{\mu}_x} \log Z \quad (4.2a)$$

and from (3.13)

$$C_{x,x'} = \bar{\mu}_x \bar{\mu}_{x'} \frac{\partial^2}{\partial \bar{\mu}_x \, \partial \bar{\mu}_{x'}} \log Z \tag{4.2b}$$

for the density–density correlation function. It is, therefore, sufficient for our purposes to evaluate the partition function of (4.1). As a first step we derive a random matrix representation of the permanent. We introduce a matrix u with statistically independent matrix elements. The distribution of these elements is restricted only by a vanishing mean

$$\langle u_{x,x'} \rangle_u = 0 \tag{4.3}$$

and the variance

$$\langle (u_{x,x'})^2 \rangle_u = 1 \tag{4.4}$$

With

$$\bar{u}_{x,x'} := (w_{x,x'})^{1/2} u_{x,x'}, \qquad u_{x,x} = 0 \tag{4.5}$$

and

$$\gamma_x := (\bar{\mu}_x)^{1/2} \tag{4.6}$$

we consider with an arbitrary parameter s

$$\begin{aligned} &\left\langle \det(s\bar{u} + \gamma) \det\left(\frac{1}{s}\bar{u} + \gamma\right) \right\rangle_u \\ &= \left\langle \left\{ \sum_{\pi_1} (-1)^{\pi_1} \prod_x [s\bar{u}_{x_1\pi_1(x)} + \gamma_x \delta_{x\pi_1(x)}] \right\} \right. \\ &\quad \times \left. \left\{ \sum_{\pi_2} (-1)^{\pi_2} \prod_x \left[\frac{1}{s}\bar{u}_{x,\pi_2(x)} + \gamma_x \delta_{x,\pi_2(x)}\right] \right\} \right\rangle_u \end{aligned} \tag{4.7}$$

Since the $\bar{u}_{x,x'}$ are statistically independent, the rhs is

$$\sum_{\pi_1,\pi_2} (-1)^{\pi_1+\pi_2} \prod_x \left\langle [s\bar{u}_{x,\pi_1(x)} + \gamma_x \delta_{x,\pi_1(x)}] \left[\frac{1}{s}\bar{u}_{x,\pi_2(x)} + \gamma_x \delta_{x,\pi_2(x)}\right] \right\rangle_u \tag{4.8}$$

where we have performed a summation over all permutations π_1, π_2 of $x \in \Lambda$. Due to $\bar{u}_{x,x} = 0$, as defined in (4.5), one has

$$\begin{aligned} &\left\langle [s\bar{u}_{x,\pi_1(x)} + \gamma_x \delta_{x,\pi_1(x)}] \left[\frac{1}{s}\bar{u}_{x,\pi_2(x)} + \gamma_x \delta_{x,\pi_2(x)}\right] \right\rangle_u \\ &= \begin{cases} \gamma_x^2 = \bar{\mu}_x & \text{if} \quad \pi_1(x) = \pi_2(x) = x \\ \langle \bar{u}_{x,\pi_1(x)} \bar{u}_{x,\pi_2(x)} \rangle & \text{if} \quad x \neq \pi_1(x), \pi_2(x) \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{4.9}$$

We notice that mixed products of γ_x and $\bar{u}_{x,\pi_j(x)}$ vanish because of the vanishing mean of $u_{x,x'}$. Finally, the statistical independence of the $u_{x,x'}$ implies

$$\langle \bar{u}_{x,\pi_1(x)} \bar{u}_{x_1,\pi_2(x)} \rangle = \delta_{\pi_1(x),\pi_2(x)} w_{x,\pi_1(x)} \tag{4.10}$$

In the product of the two determinants there appears at most the second power of any matrix element $u_{x,x'}$. Therefore, the restriction of the distribution of $u_{x,x'}$ by its mean and its variance is sufficient. Now substituting the result of the averaging into the expression in (4.7), we obtain

$$\begin{aligned} &\left\langle \det(s\bar{u}+\gamma) \det\left(\frac{1}{s}\bar{u}+\gamma\right)\right\rangle_u \\ &\quad = \sum_{\pi_1,\pi_2} (-1)^{\pi_1+\pi_2} \prod_x [w_{x,\pi_1(x)} + \gamma_x \delta_{x,\pi_1(x)}]\, \delta_{\pi_1(x),\pi_2(x)} \\ &\quad = \sum_\pi \prod_x [w_{x,\pi(x)} + \bar{\mu}_x \delta_{x,\pi(x)}] \end{aligned} \tag{4.11}$$

The rhs is again the permanent of (4.1), which is the partition function of the hard-core FL system. It is also remarkable that the rhs does not depend on the parameter s. Later we will utilize the freedom of choosing this parameter. Thus, we have expressed the partition function Z in terms of determinants of random matrices. It is well known that the partition function of free fermions is also a determinant ("fermion determinant") where $\bar{u}$ describes the motions of the free fermions on the lattice ("hopping") and γ is a chemical potential. On a more formal level we can introduce Grassmann variables $\{\psi_x^\alpha, \bar{\psi}_x^\alpha\}$ ($\alpha = 1, 2$). The partition function Z reads then by means of (4.11)

$$Z = \left\langle \int \exp\left[\sum_{x,x'} \sum_{\alpha=1}^{2} \psi_x^\alpha(\tau_\alpha \bar{u}_{x,x'} + \gamma_x \delta_{x,x'})\bar{\psi}_{x'}^\alpha\right] \prod d\psi_x^\alpha\, d\bar{\psi}_x^\alpha \right\rangle_u \tag{4.12}$$

where $\int \cdots \prod d\psi\, d\bar{\psi}$ is the usual "integration" over independent Grassmann variables[6] ψ_x^α and $\bar{\psi}_x^\alpha$. The τ_α carries the parameter s, since $\alpha = 1, 2$ are related to the first and the second determinant of (4.11), respectively: $\tau_1 = s$, $\tau_2 = 1/s$. Now $\bar{u}_{x,x'}$ appears as a random field which couples to neighboring pairs of fermion fields.

5. Colored Flux Lines

If there is only one type of FL, we have seen in the previous section that two different Grassmann (fermion) fields ψ_x^α ($\alpha = 1, 2$) yield an equivalent model. Now we can generalize the fermion representation by introducing $2N$ types of fermions. The partition function (4.12) reads for this generalized model

$$Z_N = \left\langle \int \exp\left[\sum_{x,x'} \sum_{\alpha=1}^{2N} \psi_x^\alpha \left(\frac{\tau_\alpha}{\sqrt{N}} \bar{u}_{x,x'} + \gamma_x \delta_{x,x'} \right) \bar{\psi}_{x'}^\alpha \right] \prod d\psi_x^\alpha \, d\bar{\psi}_x^\alpha \right\rangle_u \tag{5.1}$$

with $\tau_\alpha = s$ for $\alpha = 1, 2,..., N$, and $\tau_\alpha = s^{-1}$ for $\alpha = N+1$, $N+2,..., 2N$.

We have introduced a normalization $\sqrt{N}$ of the random field $\bar{u}_{x,x'}$, since we are interested in the limit $N \to \infty$. The latter leads to a soluble model due to this normalization: while the number of fermions is increasing with N, the contribution of the individual fermion lines (given by the propagator $\bar{u}_{x,x'}$) is decreasing with $N^{-1/2}$. In the next section we will investigate the limit $N \to \infty$. In the rest of this section, however, the meaning of $2N$ fermions in terms of the FLs will be discussed. For this purpose we perform the average over u in the partition function Z_N of (5.1). This is possible because the averaging can be exchanged with the integration over the Grassmann variables. It is convenient to introduce Gaussian-distributed matrix elements:

$$P(u_{x,x'}) \, du_{x,x'} = \exp\left[-\frac{1}{2} (u_{x,x'})^2 \right] \frac{du_{x,x'}}{2\sqrt{\pi}} \tag{5.2}$$

Then the averaging is just a "completing the square" in the exponent of (5.1), which leads to

$$Z_N = \int \exp\left[\frac{1}{2N} \sum_{x,x'} w_{x,x'} \sum_{\alpha,\alpha'=1}^{N} \tau_\alpha \tau_{\alpha'} \psi_x^\alpha \psi_x^{\alpha'} \bar{\psi}_{x'}^{\alpha'} \bar{\psi}_{x'}^\alpha \right.$$
$$\left. + \sum_x \gamma_x \sum_\alpha \psi_x^\alpha \bar{\psi}_x^\alpha \right] \prod d\psi_x^\alpha \, d\bar{\psi}_x^\alpha \tag{5.3}$$

We remember that only a complete product of Grassmann variables $\{\psi_x^\alpha, \bar{\psi}_x^\alpha\}$ gives a nonvanishing contribution. Therefore, the term with γ_x can be replaced by a polynom P_{2N} of

$$\tfrac{1}{2}\gamma_x^2 \sum_{\alpha,\alpha'} \psi^\alpha{}_x \bar{\psi}_x^\alpha \psi_x^{\alpha'} \bar{\psi}_x^{\alpha'} \tag{5.4}$$

such that

$$\exp\left[P_{2N}\left(1/2 \sum \gamma_x^2 \psi_x^\alpha \bar{\psi}_x^\alpha \psi_x^{\alpha'} \bar{\psi}_x^{\alpha'} \right) \right] = \exp\left(\gamma_x \sum_\alpha \psi_x^\alpha \bar{\psi}_x^\alpha \right)$$

Then we may combine the Grassmann fields as

$$\phi_x^{\alpha\alpha'} := \psi_x^\alpha \psi_x^{\alpha'}, \qquad \bar{\phi}_x^{\alpha\alpha'} := \bar{\psi}_x^{\alpha'} \bar{\psi}_x^\alpha \tag{5.5}$$

The new variables $\{\phi_x^{\alpha\alpha'}\}$ satisfy the properties (i), (ii) of the η algebra in Section 3: products of these variables are commutative and they are nilpotent. However, property (iii) is only valid for $\alpha=1$, $\alpha'=2$ (or $\alpha=2$, $\alpha'=1$), since they are constructed from the Grassmann algebra. In other words, the $\phi_x^{\alpha\alpha'}$ are independent only for different indices α and α'. The interpretation of the partition function in terms of the ϕ becomes obvious if we rewrite the summation in the exponent:

$$Z_N=\int \exp\left[\frac{1}{N}\sum_{x,x'} w_{x,x'} \sum_{\alpha>\alpha'} \tau_\alpha \tau_{\alpha'} \phi_x^{\alpha\alpha'} \bar{\phi}_{x'}^{\alpha\alpha'} + \sum_x P_{2N}\left(\bar{\mu}_x \sum_{\alpha>\alpha'} \phi_x^{\alpha\alpha'} \bar{\phi}_x^{\alpha\alpha'}\right)\right] \prod d\psi_x^\alpha \, d\bar{\psi}_x^\alpha \tag{5.6}$$

We notice that terms with $\alpha=\alpha'$ do not contribute in (5.3) and (5.4). There are $N(2N-1)$ terms in $\sum_{\alpha>\alpha'}$. We recover the original model of hard-core FLs for $N=1$, since ϕ_x^{21} can be replaced by η_x. For $N>1$ there are $N(2N-1)$ different colors of FLs, where each color (α, α') is represented by the field $\phi_x^{\alpha\alpha'}$ and its conjugate $\bar{\phi}_x^{\alpha\alpha'}$. FLs with colors (α, α') and (α'', α''') are subject to a hard-core interaction if $\alpha=\alpha''$ or $\alpha'=\alpha'''$. Otherwise they do not interact. This means that the effective interaction is weaker than in the case of a single color. Therefore, the density of FLs will be higher for the colored FLs. In particular, the exponent of the density β is equal to or less than that of the FLs with $N=1$. Due to the colors we have obtained a rescaling of the statistical weight $w_{x,x'}$ by the factor $\tau_\alpha \tau_{\alpha'}$ in (5.6). This factor is 1 for $N=1$, and is 1, s^2, or s^{-2} for $N>1$. One can rescale the field $\phi_x^{\alpha\alpha'}$ and its conjugate $\bar{\phi}_x^{\alpha\alpha'}$ by the square root of this factor in order to get a rescaled fugacity $\mu^{\alpha\alpha'}=(\tau_\alpha \tau_{\alpha'})^{-1}\bar{\mu}$. Thus, depending on the color, the FLs appear with a different fugacity (or chemical potential).

5.1. Regularization: Dynamics of Fermions

It is clear from the expression (4.11) that the partition function Z_N can vanish for certain realizations of the random matrix u. The free energy $\log Z_N$ is therefore singular. Since these singularities do not have a physical origin, we can avoid them by introducing a regularization. In the present case, it is convenient to introduce formally a relaxational dynamics for the fermions. This leads to a time-dependent Grassmann field

$$\psi_x^\alpha \rightarrow \psi_{x,t}^\alpha \tag{5.7}$$

in (5.1). Since $\psi_{x,t}^\alpha$ and $\bar{\psi}_{x,t}^\alpha$ are independent Grassmann variables, we can rename the conjugate field as

$$\bar{\psi}_{x,t}^\alpha \rightarrow \bar{\psi}_{x,t}^{\alpha+N(\text{mod } 2N)}$$

Moreover, there is a time difference operator for the dynamics such that the partition function reads

$$Z_{D,N} = \left\langle \int \exp\left\{ \sum_t \sum_\alpha \left[\sum_x \frac{1}{2} (\psi^\alpha_{x,t} \bar{\psi}^\alpha_{x,t+1} - \psi^\alpha_{x,t} \bar{\psi}^\alpha_{x,t-1}) \right.\right.\right.$$
$$\left.\left.\left. + \sum_{x,x'} \psi^\alpha_{x,t} \left(\frac{\tau_\alpha}{\sqrt{N}} \bar{u}^\alpha_{x,x'} + \gamma_x \delta_{x,x'} \right) \bar{\psi}^{\alpha+N(\mathrm{mod}\,2N)}_{x',t} \right] \right\} \prod_{x,t,\alpha} d\psi^\alpha_{x,t}\, d\bar{\psi}^\alpha_{x,t} \right\rangle_u \tag{5.8}$$

with

$$\bar{u}^1 = \cdots = \bar{u}^N = \bar{u},\ \bar{u}^{N+1} = \cdots = \bar{u}^{2N} = \bar{u}^T$$

The dynamics enters in a simple form because we consider the random matrix elements $\bar{u}_{x,x'}$ as static (time-independent). Thus, we can diagonalize the time dependence by a Fourier transformation. In terms of the corresponding Matsubara frequency ε the partition function is a product over ε-dependent partition functions:

$$Z_{D,N} = \prod_\varepsilon Z_{\varepsilon,N} \tag{5.9}$$

with

$$Z_{\varepsilon,N} = \left\langle \int \exp\left\{ \sum_{x,x'} \sum_\alpha \left[\psi^\alpha_{x,\varepsilon} i\varepsilon \delta_{x,x'} \bar{\psi}^\alpha_{x',\varepsilon} \right.\right.\right.$$
$$\left.\left.\left. + \psi^\alpha_{x,\varepsilon} \left(\gamma_x \delta_{x,x'} + \frac{\tau_\alpha}{\sqrt{N}} \bar{u}^\alpha_{x,x'} \right) \bar{\psi}^{\alpha+N(\mathrm{mod}\,2N)}_{x',\varepsilon} \right] \right\} \prod_{x,\alpha} d\psi^\alpha_{x,\varepsilon}\, d\bar{\psi}^\alpha_{x,\varepsilon} \right\rangle_u \tag{5.10}$$

In order to understand the regularization, we integrate over the Grassmann variables. This leads to

$$Z_{\varepsilon,N} = \left\langle \left[\det \begin{pmatrix} i\varepsilon & \frac{s}{\sqrt{N}} \bar{u} + \gamma \\ \frac{s}{\sqrt{N}} \bar{u}^T + \gamma & i\varepsilon \end{pmatrix} \det \begin{pmatrix} i\varepsilon & \frac{1}{s\sqrt{N}} \bar{u} + \gamma \\ \frac{1}{s\sqrt{N}} \bar{u}^T + \gamma & i\varepsilon \end{pmatrix} \right]^{N/2} \right\rangle_u \tag{5.11}$$

The Matsubara frequency from the relaxational term adds as an imaginary term to the Hermetian matrix

$$\begin{pmatrix} 0 & \frac{s}{\sqrt{N}} \bar{u} + \gamma \\ \frac{s}{\sqrt{N}} \bar{u}^T + \gamma & 0 \end{pmatrix} \tag{5.12}$$

Therefore, the determinants in $Z_{\varepsilon,N}$ do not vanish for any realization of u. On the other hand, the long-time (low-frequency) limit gives the original model as defined before:

$$\lim_{\varepsilon\to 0} Z_{\varepsilon,N} = \left\langle \left[\det\left(\frac{s}{\sqrt{N}}\bar{u}+\gamma\right) \det\left(\frac{1}{s\sqrt{N}}\bar{u}+\gamma\right)\right]^N \right\rangle_u \tag{5.13}$$

Thus, the frequency term $i\varepsilon$ is a regularization of the model. Of course, it would not be needed if we treat the model on a finite lattice with appropriate boundary conditions, since the partition function cannot be zero then. However, this requires a careful discussion of the finite lattice. The regularization allows a more liberal study, because only for the expressions we consider at the end do we have to be careful with the existence of the static limit ε. We will see that this limit and the thermodynamic limit cannot be interchanged.

6. THE $N \to \infty$ LIMIT

In order to understand the colored FL model of the previous section we have performed the average with respect to the random matrix u in Z_N of (5.1). To discuss the asymptotic behavior of Z_N for large N, it is more convenient to integrate over the Grassmann field ψ_x^α first. Then we obtain the result of Section 5.1, Eq. (5.11),

$$Z_{\varepsilon,N} = \left\langle \left[\det\begin{pmatrix} i\varepsilon & \frac{s}{\sqrt{N}}\bar{u}+\gamma \\ \frac{s}{\sqrt{N}}\bar{u}^T+\gamma & i\varepsilon \end{pmatrix} \times \det\begin{pmatrix} i\varepsilon & \frac{1}{s\sqrt{N}}\bar{u}+\gamma \\ \frac{1}{s\sqrt{N}}\bar{u}^T+\gamma & i\varepsilon \end{pmatrix} \right]^{N/2} \right\rangle_u$$

since $\{\psi_x^\alpha\}$ are independent Grassmann variables. Using again Gaussian-distributed matrix elements $u_{x,x'}$ as defined in (5.2), we can rescale these matrix elements by $N^{-1/2}$. Then we get

$$Z_{\varepsilon,N} = \int \left[\det\begin{pmatrix} i\varepsilon & s\bar{u}+\gamma \\ s\bar{u}^T+\gamma & i\varepsilon \end{pmatrix} \det\begin{pmatrix} i\varepsilon & (1/s)\bar{u}+\gamma \\ (1/s)\bar{u}^T+\gamma & i\varepsilon \end{pmatrix}\right]^{N/2} \times \prod_{x,x'}{}' \frac{1}{2}\left(\frac{N}{\pi}\right)^{1/2} \exp\left[-\frac{N}{2}(u_{x,x'})^2\right] du_{x,x'} \tag{6.1}$$

The product $\prod'$ goes over pairs $x, x' = (r, x), (r', z')$ with $(r', z') = (r \pm e_j, z+1)$ (for $j=1, 2,..., d-1$) according to our definition of the FL element by $w_{x,x'}$ in (3.4). The form of the partition function suggests a saddle-point approximation: for large N the integral over the matrix elements is dominanted by the maxima of the integrand. It will turn out that there is only a single maximum for translational invariant realizations of $u_{x,x'}$. We obtain a $1/N$ expansion from the expansion around this maximum.

The extrema (or saddle points) of the partition function can be evaluated from the saddle-point equation

$$\frac{\partial}{\partial u_{x,x'}} \left\{ \frac{1}{2} (u_{x,x'})^2 - \frac{1}{2} \log \det \begin{pmatrix} i\varepsilon & s\bar{u}+\gamma \\ s\bar{u}^T+\gamma & i\varepsilon \end{pmatrix} \right.$$

$$\left. - \frac{1}{2} \log \det \begin{pmatrix} i\varepsilon & \frac{1}{s}\bar{u}+\gamma \\ \frac{1}{s}\bar{u}^T+\gamma & i\varepsilon \end{pmatrix} \right\} = 0 \qquad (6.2)$$

The further discussion will be restricted to translational invariant solutions u_o of (6.2):

$$(u_o)_{x,x'} = (u_o)_{x'-x} \qquad (6.3)$$

which is suggested by the exact solution of the two-dimensional model. The Fourier components of the solutions can be written as

$$\tilde{u}_o = [2(d-1)]^{1/2} \sigma \kappa e^{i\omega} \qquad (6.4)$$

with

$$\kappa = \frac{1}{d-1} \sum_{j=1}^{d-1} \cos k_j \qquad (6.5)$$

for

$$-\pi < \omega, k_j < \pi \qquad (6.6)$$

Then σ must be determined by the saddle-point equation. We obtain from (6.2) for $\varepsilon \sim 0$

$$\sigma^2 \sim \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left[\theta(s^2\sigma^2\kappa^2 - \gamma^2) + \theta\left(\frac{\sigma^2}{s^2}\kappa^2 - \gamma^2\right) \right] \frac{dk_1}{2\pi} \cdots \frac{dk_{d-1}}{2\pi} \qquad (6.7)$$

where θ is the step function. The integral can be rewritten by means of the density

$$\rho_{d-1}(\kappa)=\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\operatorname{Im}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left[\frac{1}{d-1}\sum_{j=1}^{d-1}\cos k_j-\kappa-i\varepsilon\right]^{-1}\frac{dk_1}{2\pi}\cdots\frac{dk_{d-1}}{2\pi} \tag{6.8}$$

The saddle-point condition for σ then reads

$$\sigma^2\sim 2\int_{|\gamma/s\sigma|}^{1}\rho_{d-1}(\kappa)\,d\kappa+2\int_{|s\gamma/\sigma|}^{1}\rho^{d-1}(\kappa)\,d\kappa \tag{6.9}$$

Now we take advantage of the fact that s is a free parameter which can be chosen as

$$s=\sigma \tag{6.10}$$

This leads, from (6.9), to

$$\sigma^2\sim 2\int_{|\gamma/\sigma^2|}^{1}\rho_{d-1}(\kappa)\,d\kappa+2\int_{|\gamma|}^{1}\rho_{d-1}(\kappa)\,d\kappa \tag{6.11}$$

$\gamma^2=\bar{\mu}$ is a parameter of the FL model describing the weight (fugacity) of empty sites on the lattice. Since the density $\rho_{d-1}(\kappa)$ vanishes for $|\kappa|>1$, we obtain

$$\sigma^2\sim 2\int_{|\gamma/\sigma^2|}^{1}\rho_{d-1}(\kappa)\,d\kappa\qquad\text{for}\quad |\gamma|>1 \tag{6.12}$$

Furthermore, $\rho_{d-1}(\kappa)\geqslant 0$ and

$$2\int_{0}^{1}\rho_{d-1}(\kappa)\,d\kappa=1 \tag{6.13}$$

This implies, for $|\gamma|>1$,

$$0\leqslant 2\int_{|\gamma/\sigma^2|}^{1}\rho^{d-1}(\kappa)\,d\kappa\leqslant 2\int_{1/\sigma^2}^{1}\rho^{d-1}(\kappa)\,d\kappa\leqslant 1 \tag{6.14}$$

and due to (6.12), $\sigma^2\leqslant 1$. Consequently, with (6.12), $\sigma^2=0$ if $\varepsilon=0$. Therefore, $\gamma>1$ describes the Meissner phase (no FLs). On the other hand, $\sigma^2=2$ for $\gamma=0$. The behavior of $\sigma^2(\gamma)$ depends on the dimension d for $0<|\gamma|\leqslant 1$. In particular, we have

$$\sigma^2\sim 2\int_{|\gamma|}^{1}\rho_{d-1}(\kappa)\,d\kappa\qquad\text{for}\quad \sigma^2<|\gamma| \tag{6.15}$$

σ^2 can be evaluated easily from (6.11) for $d=2$, 3, because we have

$$\rho_1(\kappa)=\frac{1}{\pi}(1-\kappa^2)^{-1/2}\,\theta(1-\kappa^2) \tag{6.16}$$

$$\rho_2(\kappa)=\frac{4}{\pi^2}\frac{1}{1+\kappa}K\left(\left(\frac{1-\kappa}{1+\kappa}\right)^2\right)\theta(1-\kappa^2) \tag{6.17}$$

where $K(y)$ is the complete elliptic integral.[8] The densities of higher dimension have the asymptotic behavior

$$\rho_{d-1}(\kappa)\sim \text{const}\cdot(1-|\kappa|)^{(d-3)/2} \qquad \text{for} \quad |\kappa|\sim 1 \tag{6.18}$$

We are now in a position to evaluate physical quantities which can be expressed in terms of the free energy of the infinite system ($\Lambda\uparrow Z^d$),

$$\begin{aligned} F_{\varepsilon,N}&=\lim_{\Lambda\uparrow Z^d}\frac{1}{N|\Lambda|}\log Z_{\varepsilon,N}\\ &=\frac{1}{2}\int\Bigg[\log(\varepsilon^2+|\sigma\bar{u}_{o,k}+\gamma|^2)\\ &\quad+\log\left(\varepsilon^2+\frac{1}{\sigma}\bar{u}_{o,k}+\gamma|^2\right)\Bigg]\frac{d\omega}{2\pi}\frac{d^{d-1}k}{(2\pi)^{d-1}}+O(N^{-1}) \end{aligned} \tag{6.19}$$

The density of FLs reads then, according to (4.2a),

$$n(\gamma)=1-\lim_{\varepsilon\downarrow 0}\bar{\mu}\frac{\partial}{\partial\bar{\mu}}F_{\varepsilon,N}=1-\lim_{\varepsilon\downarrow 0}\frac{\gamma}{2}\frac{\partial}{\partial\gamma}F_{\varepsilon,N} \tag{6.20}$$

and, with (6.19),

$$\begin{aligned} n(\gamma)=1-\lim_{\varepsilon\downarrow 0}\frac{\gamma}{2}\int\left[\frac{\sigma^2\kappa e^{i\omega}+\gamma}{\varepsilon^2+|\sigma^2\kappa e^{i\omega}+\gamma|^2}+\frac{\kappa e^{i\omega}+\gamma}{\varepsilon^2+|\kappa e^{i\omega}+\gamma|^2}\right]\frac{d\omega}{2\pi}\frac{d^{d-1}k}{(2\pi)^{d-1}}\\ +O(N^{-1}) \end{aligned} \tag{6.21}$$

The ω-integration has been carried out in Appendix A, with the result

$$\eta(\gamma)=1-\int_0^{|\gamma/\sigma^2|}\rho^{d-1}(\kappa)\,d\kappa-\int_0^{|\gamma|}\rho_{d-1}(\kappa)+O(N^{-1}) \tag{6.22}$$

For $\sigma^2<|\gamma|$ we get

$$n(\gamma)=\int_{|\gamma|}^1\rho^{d-1}(\kappa)\,d\kappa+O(N^{-1}) \tag{6.23}$$

We will see in the next section that the terms $O(N^{-1})$ vanish if $|\gamma| \to 1$. Thus, the density of FLs vanishes for $|\gamma| \geqslant 1$; i.e., $|\gamma| = 1$ (or $\bar{\mu} = 1$) is the critical point of the Abrikosov–Meissner transition. The asymptotic behavior of the density is, in saddle-point approximation,

$$n(\gamma) \sim n_0(1 - |\gamma|)^{d-1)/2} \qquad \text{for} \quad |\gamma| \leqslant 1 \tag{6.24}$$

The coefficient n_0 can be logarithmically divergent (e.g., for $d = 3$), as we see from the expression in (6.23). The power law is in agreement with a general random walk argument by Fisher[1] for $d \leqslant 3$.

From the free energy we can also derive the density–density correlation function according to (4.2b). A simple calculation leads for $\sigma^2 < |\gamma|$ to the longitudinal correlation function

$$C_{\|,z} = \frac{\gamma^2}{2} G_{12;0,z} G_{12;0,-z} + O(N^{-1}) \tag{6.25}$$

With the Green's function defined in Appendix A,

$$G_{12;-r,-z} = \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{(\kappa e^{i\omega} + \gamma) e^{i\omega z + ik \cdot r}}{\varepsilon^2 + |\kappa e^{i\omega} + \gamma|^2} \, d\omega \, d^{d-1}k$$

we obtain from Appendix A

$$C_{\|,z} = \frac{1}{2} \int_0^{|\gamma|} \kappa^z \rho_{d-1}(\kappa) \, d\kappa \int_{|\gamma|}^1 \kappa^{-z} \rho_{d-1}(\kappa) \, d\kappa + O(N^{-1}) \tag{6.26}$$

Pulling the densities out of the integrals, we find an exponential decay of the form

$$C_{\|,z} = \frac{1}{2} \rho^{d-1}(\kappa_1) \, \rho_{d-1}(\kappa_2) \gamma^2 \frac{1 - |\gamma|^{z-1}}{z^2 - 1} + O(N^{-1}) \tag{6.27}$$

with some values κ_1, κ_2:

$$0 \leqslant \kappa_1 \leqslant |\gamma| \qquad \text{and} \qquad |\gamma| \leqslant \kappa_2 \leqslant 1 \tag{6.28}$$

Thus, there is a correlation length

$$\xi_{\|} \sim (1 - |\gamma|)^{-1} \qquad \text{for} \quad |\gamma| \leqslant 1 \tag{6.29}$$

The transverse density–density correlations are isotropic. Therefore, it is sufficient to consider the x_1 direction for $\sigma^2 < |\gamma|$:

$$C_{\perp,r_1} = \frac{\gamma^2}{2} G_{12;(r_1,0,\ldots,0),0} G_{12;(-r_1,0,\ldots,0),0} + O(N^{-1}) \tag{6.30}$$

Again from Appendix A we obtain

$$C_{\perp, r_1} = \frac{1}{2} \left| \int e^{ik_1 r_1} \theta(\gamma^2 - \kappa^2) \frac{d^{d-1}k}{(2\pi)^{d-1}} \right|^2 + O(N^{-1}) \tag{6.31}$$

The $(d-1)$-dimensional k-integration can be reduced to a one-dimensional integration of the form

$$C_{\perp, r_1} = \frac{2}{\pi^2 r_1^2} \left\{ \int_{[(d-1)\gamma - 1]/(d-2)}^{1} \rho_{d-2}(t) \sin[r_1/\xi_\perp(d, \gamma, t)]\, dt \right\}^2 + O(N^{-1}) \tag{6.32}$$

with a characteristic length scale

$$\xi_\perp(d, \gamma, t) = \{\arccos[(d-1)\gamma - (d-2)t]\}^{-1} \tag{6.33}$$

If we approach the critical point $|\gamma| = 1$, the asymptotic behavior of this length scale is, for $t \sim 1$,

$$\xi_\perp \sim [2(d-1)(1-|\gamma|)]^{-1/2} \tag{6.34}$$

7. FLUCTUATIONS AROUND THE SADDLE POINT: 1/*N* EXPANSION

The saddle-point approximation, which is also the limit $N \to \infty$ of the colored FL model, is a model of free fermions. It describes exactly the statistics of FLs for $d=2$, $N=1$ if we substitute $\gamma \to \gamma^2 = \bar{\mu}$. Therefore, the fluctuations around the saddle-point solution cannot contribute to the model in a relevant manner. The situation might be different in $d > 2$: fluctuations of the random field $u_{x,x'}$ could lead to a modification of the saddle-point result, since they are responsible for the difference from the free fermions due to the entanglement of FLs. Nevertheless, we will show in this section that the fluctuations are irrelevant with respect to the Abrikosov–Meissner transition in any dimension d. For this purpose we consider again the partition function $Z_{\varepsilon,N}$ of (6.1), introducing a free energy $F_{\varepsilon,N}$:

$$Z_{\varepsilon,N} = \int \exp(-F_{\varepsilon,N}) \prod_{x,x'}{}' \frac{1}{2} \left(\frac{N}{\pi}\right)^{1/2} du_{x,x'} \tag{7.1}$$

with

$$F_{\varepsilon,N} = \frac{N}{2} \left\{ \sum_{x,x'} [(u_0 + \delta u)_{x,x'}]^2 - \log\det \begin{pmatrix} i\varepsilon & \sigma(\bar{u}_o + \delta\bar{u}) + \gamma \\ \sigma(\bar{u}_o^T + \delta\bar{u}^T) + \gamma & i\varepsilon \end{pmatrix} \right.$$
$$\left. - \log\det \begin{pmatrix} i\varepsilon & (1/\sigma)(\bar{u}_o + \delta\bar{u}) + \gamma \\ (1/\sigma)(\bar{u}_o^T + \delta\bar{u}^T) + \gamma & i\varepsilon \end{pmatrix} \right\} \tag{7.2}$$

The random matrix is here separated into the saddle-point solution u_o and the fluctuation δu around u_o. The expansion of $F_{\varepsilon,N}$ in powers of the fluctuations δu is apparently a $1/N$ expansion. This becomes obvious when we rescale δu by $\sqrt{N}$: the fluctuations appear with $N^{-1/2}$ in the determinants of $F_{\varepsilon,N}$. In particular, the second-order contribution (Gaussian fluctuations) is $O(N^0)$ then. Thus, at least for large values of N, we can restrict the investigation of the fluctations to Gaussian fluctuations of $F_{\varepsilon,N}$:

$$\begin{aligned}G_{\text{Gauss}}=&\frac{1}{2}\sum_{x,x'}(\delta u_{x,x'})^2\\&+\frac{1}{4\sigma^2}\sum_{x,\dots,x'''}\mathrm{Tr}_2\left[G_{x,x'}\begin{pmatrix}0&\delta\bar{u}\\\delta\bar{u}^T&0\end{pmatrix}_{x',x''}G_{x'',x''}\begin{pmatrix}0&\delta\bar{u}\\\delta\bar{u}^T&0\end{pmatrix}_{x''',x}\right]\\&+\frac{\sigma^2}{4}\sum_{x,\dots,x'''}\mathrm{Tr}_2\left[G'_{x,x'}\begin{pmatrix}0&\delta\bar{u}\\\delta\bar{u}^T&0\end{pmatrix}_{x',x''}G'_{x'',x'''}\begin{pmatrix}0&\delta\bar{u}\\\delta\bar{u}^T&0\end{pmatrix}_{x''',x}\right]\end{aligned} \tag{7.3}$$

where

$$G=\begin{pmatrix}i\varepsilon&(1/\sigma)\bar{u}_o+\gamma\\(1/\sigma)\bar{u}_o^T+\gamma&i\varepsilon\end{pmatrix}^{-1} \tag{7.4a}$$

$$G'=\begin{pmatrix}i\varepsilon&\sigma\bar{u}_o+\gamma\\\sigma\bar{u}_o^T+\gamma&i\varepsilon\end{pmatrix}^{-1} \tag{7.4b}$$

We notive here that the expansion of the second term in (7.2) vanishes for any order if $\sigma^2<|\gamma|$. This feature of the expansion is related to the fact that the FLs and, therefore, $u_{x,x'}$ are directed in the z direction. A discussion of this property is given in Appendix B. Since we have already utilized the simplification of the saddle point for $\sigma^2<|\gamma|$, we will also restrict the investigation of the $1/N$ expansion to this very region of the parameter γ. The Gaussian fluctuations are then reduced to the first and the second terms of F_{Gauss} in (7.3). Moreover, we introduce the field

$$\varphi_{x,j}:=\delta\bar{u}_{x,x+e'_j}\qquad(e'_j=e_j+e_z),\qquad j=1,2,\dots,2(d-1) \tag{7.5}$$

since the matrix elements $u_{x,x'}$ are nonzero only for $x'=x\pm e'_j$. Then we may diagonalize F_{Gauss} by a Fourier transformation because G and G' are translational invariant:

$$F_{\text{Gauss}}=\sum_{j,l=1}^{2(d-1)}\frac{1}{(2\pi)^d}\int\tilde{\varphi}_{\mathbf{k},l}I_{\mathbf{k};l,j}\tilde{\varphi}_{-\mathbf{k},j}d^d\mathbf{k} \tag{7.6}$$

with the wave vector $\mathbf{k}=(k_1,\dots,k_{d-1},\omega)$. The $2(d-1)\times 2(d-1)$ stability matrix is

$$I_{\mathbf{k};l,j}=(d-1)\delta_{l,j}+\frac{1}{2\sigma^2}\,\mathrm{Re}\,\frac{1}{(2\pi)^d}\int A_{\mathbf{k}'-\mathbf{k}}e^{i(\mathbf{k}'-\mathbf{k})\cdot e_l'}A_{\mathbf{k}'}e^{i\mathbf{k}'\cdot e_j'}\,d^d\mathbf{k}'$$
$$+\frac{1}{2\sigma^2}\frac{1}{(2\pi)^d}\int C_{\mathbf{k}'-\mathbf{k}}\,e^{i(\mathbf{k}-\mathbf{k}')\cdot(e_j'-e_l')}\,C_{\mathbf{k}'}d^d\mathbf{k}' \tag{7.7}$$

with

$$A_{\mathbf{k}}=\frac{\kappa e^{-i\omega}+\gamma}{\varepsilon^2+|\kappa e^{i\omega}+\gamma|^2} \tag{7.8a}$$

$$C_{\mathbf{k}}=\frac{i\varepsilon}{\varepsilon^2+|\kappa e^{i\omega}+\gamma|^2} \tag{7.8b}$$

Here we must be careful with the thermodynamic and the static limit ($\varepsilon\to 0$) because they do not interchange. For a finite system, where the integrals in (7.6), (7.7) are replaced by sums, we consider the limit $\varepsilon\to 0$. Then we can return to the integrals due to the thermodynamic limit. In this article we will not discuss the dynamic system where the limits appear in reversed order.

The stability matrix can now be evaluated by performing the $\mathbf{k}$-integration. In particular, we find on long scales (i.e., $\mathbf{k}=0$)

$$I_{0;j,l}=(d-1)\delta_{j,l}+\frac{1}{2\sigma^2}\frac{1}{(2\pi)^{d-1}}\int\frac{\theta(\kappa^2-\gamma^2)}{\kappa^2}e^{ik\cdot(e_l+e_j)}\,d^{d-1}k \tag{7.9}$$

The second term, which is the contribution of the log det term to the Gaussian fluctuations, vanishes for $|\gamma|>1$ since $\kappa^2\leqslant 1$. This corresponds to the fact that there are no FLs (Meissner phase) for $\bar{\mu}=\gamma^2>1$. On the other hand, we can evaluate the eigenvalues of I_0. For $d=2$ we obtain

$$I_{0;j,j}=1+\frac{1}{2\sigma^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\theta(\kappa^2-\gamma^2)\frac{2\kappa^2-1}{\kappa^2}\,dk \tag{7.10a}$$

and

$$I_{0;1,2}=I_{0;2,1}=\frac{1}{2\sigma^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\theta(\kappa^2-\gamma^2)\frac{1}{\kappa^2}\,dk \tag{7.10b}$$

With σ given in (6.7), the eigenvalues of I_0 read

$$\lambda_1 = 2 \tag{7.11a}$$

and

$$\lambda_2 = 1 + \frac{1}{\sigma^2} \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta(\kappa^2 - \gamma^2) \frac{\kappa^2 - 1}{\kappa^2} \, dk \sim 2 - \frac{1}{\gamma^2} \qquad \text{for} \quad |\gamma| \sim 1 \tag{7.11b}$$

Both eigenvalues are positive for $|\gamma| \sim 1$; i.e., there are no large fluctuations near the critical point. In higher dimensions we can estimate the eigenvalues of I_0 (see Appendix C) as

$$\lambda_j \geqslant d - 1 + \frac{1}{\gamma^2} \left[\frac{1}{2} - (d-1) \left(\frac{2d-3}{2d-2} \right)^{1/2} \right] \tag{7.12}$$

which is also positive near the critical point. This means that the critical properties at the Abrikosov–Meissner transition are not affected by the critical fluctuations for $d \geqslant 2$, and there the saddle-point solution is a good approximation.

8. CONCLUSION

In this article we have discussed the statistics of interacting FLs on a lattice which are subject to thermal fluctuations. The model we considered is very simple: we restricted the interaction to a hard-core repulsion and kept the length of the FLs as fixed. Nevertheless, this might be a good description of the FLs in superconductors with short coherence length (e.g., high-T_c superconductors), directed polymers,[2] or, in two dimensions, domain walls on surfaces near a commensurate–incommensurate phase transition.[7] FLs, usually considered as Bose world lines, have been studied here as a fermion problem: it turned out from our calculation that a system of hard-core FLs is equivalent to a system of free fermions coupled to a random field. Of course, such a model cannot be solved exactly. However, we can start from the free fermions and treat the fluctuations due to the random field as a perturbation. If we consider the more general case of colored FLs, it is possible to solve the limit of infinitely many different colors. The latter, as well as the two-dimensional system with one color, are exactly described by free fermions. The $N \to \infty$ limit appears as a saddle-point problem. On the other hand, fluctuations around this saddle point are related to an $1/N$ expansion, such that results for finite values of N can be obtained from this expansion. Fortunately, the fluctuations are only irrelevant perturbations with respect to the Abrikosov–Meissner transition where the density of FLs is vanishing. However, in the literature there are alternative approaches for the case $N=1$. As already mentioned in the introduction, the FL problem can be treated in terms of interacting bosons. It is argued that the interaction is a relevant perturbation only for $d<3$. Therefore, the mean-field result is a good approximation for $d>3$. It yields a linear density of FL instead of the power law in (6.24). This was also found from a transfer matrix calculation[9] which is based on the statistics of two FLs. Thus, it seems to be impossible to continue from $N\sim\infty$ to $N=1$. This observation deserves further investigation. For instance, it is possible that there is a critical value N_c which separates the validity of the $1/N$ expansion from the validity of the free boson theory. Nevertheless, both approaches are in agreement for $d\leqslant 3$. In particular, we get for the density of FLs the asymptotic behavior near the Abrikosov–Meissner transition as [Eq. (6.24)]

$$n(\gamma)\sim n_0(1-|\gamma|)^{(d-1)/2} \qquad \text{for} \quad |\gamma|\leqslant 1$$

where the coefficient n_0 is logarithmically divergent for $d=3$. For the density–density correlation parallel to the external magnetic field (z direction) we find [Eq. (6.27)]

$$C_{\|,z}=\frac{1}{2}\rho^{d-1}(\kappa_1)\,\rho_{d-1}(\kappa_2)\gamma^2\frac{1-|\gamma|^{z-1}}{z^2-1}+O(N^{-1})$$

with some values κ_1, κ_2 [see (6.28)]:

$$0\leqslant\kappa_1\leqslant|\gamma| \qquad \text{and} \qquad |\gamma|\leqslant\kappa_2\leqslant 1$$

The correlation perpendicular to the directions reads [Eq. (6.32)]

$$C_{\perp,r_1}=\frac{2}{\pi^2 r_1^2}\left\{\int_{[(d-1)\gamma-1]/(d-2)}^{1}\rho^{d-2}(t)\sin[r_1/\xi_\perp(d,\gamma,t)]\,dt\right\}^2+O(N^{-1})$$

This implies a longitudinal correlation length $\xi_\|$ and a transverse correlation length $\xi_\perp$. The asymptotic behavior of these lengths near the Abrikosov–Meissner transition ($|\gamma|\sim 1$) is [see (6.29)]

$$\xi_\|\sim(1-|\gamma|)^{-1}$$

and [see (6.34)]

$$\xi_\perp\sim[2(d-1)(1-|\gamma|)]^{-1/2}$$

The fluctuations of the random field coupled to the free fermions are related to the effect of entanglements of FLs. The hard-core interaction among the FLs is described by the fermion character of the field theory. In the boson approach,[4] on the other hand, the effect of the entanglement on the statistics is correctly described already by the free bosons. However, the interaction among the FLs can only be treated perturbatively. It turns out from the $1/N$ expansion in the free-fermion approach that entanglement effects (i.e., fluctuations of the random field) are irrelevant. In Section 2.2 we have also discussed the case of noninteracting FLs with the boundary condition that at most one FL per site can start at the bottom. This boundary condition, which is a hard-core interaction on the surface, guarantees a noncollapsing system. A simple calculations shows that, for instance, the density of FLs has a peculiar behavior: in contrast to the power law in the presence of interactions, we find here a discontinuity (step) at the transition point. This implies that the interaction is a relevant perturbation. Therefore, the free-fermion saddle point is the appropriate description of interacting FLs, since it takes the relevant perturbation into account. This observation has been utilized for the investigation of the effect of random impurities (quenched local potentials) where we started from the free-fermion saddle point.[10]

APPENDIX A: MATRIX ELEMENTS IN THE 1/N EXPANSION

The following matrix elements occur in the $1/N$ expansion of the fermion determinants:

$$G_{x'-x}=\begin{pmatrix} i\varepsilon & \bar{u}_o+\gamma \\ \bar{u}_o^T+\gamma & i\varepsilon \end{pmatrix}^{-1}_{x,x'} \tag{A.1}$$

Consequently, we have to evaluate in terms of its Fourier components:

$$G_{12;-r,-z}=\frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{(\kappa e^{i\omega}+\gamma)e^{i\omega z+ik\cdot r}}{\varepsilon^2+|\kappa e^{i\omega}+\gamma|^2}\,d\omega\,d^{d-1}k \tag{A.2}$$

$$G_{11;-r,-z}=\frac{i\varepsilon}{(2\pi)^d}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{e^{i\omega z+ik\cdot r}}{\varepsilon^2+|\kappa e^{i\omega}+\gamma|^2}\,d\omega\,d^{d-1}k \tag{A.3}$$

The ω integration can be performed easily. For this purpose it is convenient to introduce $y=e^{i\omega}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(e^{i\omega})e^{i\omega z}}{\varepsilon^2+|\kappa e^{i\omega}+\gamma|^2}\,d\omega=\frac{1}{2\pi i}\oint\frac{f(y)\,y^z}{\gamma\kappa(y-y_-)(y-y_+)}\,dy \tag{A.4}$$

with the poles of the integrand

$$y_\pm=-\frac{1}{2\kappa\gamma}\{\varepsilon^2+\kappa^2+\gamma^2\pm[(\varepsilon^2+\kappa^2+\gamma^2)^2-4\kappa^2\gamma^2]^{1/2}\} \tag{A.5}$$

In particular, we have

$$y_+ y_- = 1 \tag{A.6}$$

and

$$|y_+| > 1 \tag{A.7}$$

The latter follows from

$$|y_+| - 1 = -y_+ - 1 = \frac{1}{2\kappa\gamma}\{\varepsilon^2 + (\kappa-\gamma)^2 + [(\varepsilon^2+\kappa^2+\gamma^2)^2 - 4\kappa^2\gamma^2]^{1/2}\} > 0 \tag{A.8}$$

The integration in (A.4) is performed along the unit circle. Therefore, only the pole at y_- contributes to the integral. This yields eventually, for $\varepsilon \sim 0$,

$$\frac{1}{2\pi i}\oint \frac{(\kappa y+\gamma)\, y^z}{\gamma\kappa(y-y_-)(y-y_+)}\, dy \sim (-\gamma)^{-z-1}\kappa^z \begin{cases} -\theta(\gamma^2-\kappa^2), & z \geqslant 0 \\ \theta(\kappa^2-\gamma^2), & z < 0 \end{cases} \tag{A.9}$$

and

$$\frac{\varepsilon}{2\pi i}\int \frac{y^z}{\gamma\kappa(y-y_-)(y-y_+)}\, dy \sim 0 \tag{A.10}$$

(A.9) means in particular that matrix elements $G_{12;-r,-z}$ vanish for $z<0$ if $\gamma^2 > 1$, because of $\kappa^2 \leqslant 1$.

APPENDIX B. VANISHING TERMS IN THE 1/*N* EXPANSION

As discussed in Section 7, the $1/N$ expansion consists in the expansion of the type

$$\begin{aligned} &N \log\det \begin{pmatrix} i\varepsilon & (1/s)(\bar{u}_o + N^{-1/2}\delta\bar{u}) + \gamma \\ (1/s)(\bar{u}_o^T + N^{-1/2}\delta\bar{u}^T) - \gamma & i\varepsilon \end{pmatrix} \\ &= N \log\det \begin{pmatrix} i\varepsilon & (1/s)\bar{u}_o + \gamma \\ (1/s)\bar{u}_o^T + \gamma & i\varepsilon \end{pmatrix} \\ &\quad + N \sum_{l \geqslant 1} \frac{(-s^{-1}N^{-1/2})^l}{l} \operatorname{Tr}\left\{\left[\begin{pmatrix} i\varepsilon & (1/s)\bar{u}_o + \gamma \\ (1/s)\bar{u}_o^T + \gamma & i\varepsilon \end{pmatrix}^{-1} \begin{pmatrix} 0 & \delta\bar{u} \\ \delta\bar{u}^T & 0 \end{pmatrix}\right]^l\right\} \end{aligned} \tag{B.1}$$

We obtain, for $\varepsilon \sim 0$,

$$\begin{aligned} &\operatorname{Tr}\left\{\left[\begin{pmatrix} i\varepsilon & (1/s)\bar{u}_o + \gamma \\ (1/s)\bar{u}_o^T + \gamma & i\varepsilon \end{pmatrix}^{-1} \begin{pmatrix} 0 & \delta\bar{u} \\ \delta\bar{u}^T & 0 \end{pmatrix}\right]^l\right\} \\ &\quad \sim \operatorname{Tr}[(G_{12}\delta\bar{u}^T)^l] + \operatorname{Tr}[(G_{12}^T\delta\bar{u})^l] = 2\operatorname{Tr}[(G_{12}\delta\bar{u}^T)^l] \end{aligned} \tag{B.2}$$

where G_{12} and G_{11} were defined in Appendix A. Now we consider the matrix elements on the rhs of (B.2) with respect to the z dependence. We remember that

$$\delta\bar{u}_{r,z,r',z'} \neq 0 \qquad \text{only for} \quad z' = z+1 \tag{B.3}$$

Thus, in the expression

$$\begin{aligned}\mathrm{Tr}[(G_{12}\,\delta\bar{u}^T)^l] &= \sum (G_{12})_{z,\bar{z}}(\delta\bar{u}^T)_{\bar{z},z'}(G_{12})_{z',\bar{z}'}\cdots(G_{12})_{z^{(l-1)},\bar{z}^{(l-1)}}(\delta\bar{u}^T)_{\bar{z}^{(l-1)},z}\\ &= \sum (G_{12})_{\bar{z}-z}(\delta\bar{u})_{z',\bar{z}}(G_{12})_{\bar{z}'-z'}\cdots(G_{12})_{\bar{z}^{(l-1)}-z^{(l-1)}}(\delta\bar{u})_{z,\bar{z}^{(l-1)}}\end{aligned} \tag{B.4}$$

one finds

$$z^{(k+1)} = \bar{z}^{(k)} - 1 \tag{B.5}$$

due to $\delta\bar{u}$. Therefore, there must be at least one pair $z^{(k)}$, $\bar{z}^{(k)}$ with

$$\bar{z}^{(k)} > z^{(k)} \tag{B.6}$$

because of the trace. From Appendix A we know that, for $z > 0$,

$$G_{12,z} \sim 0 \qquad \text{if} \quad \gamma^2 > 1 \tag{B.7}$$

As a consequence, the expansion terms vanish for $\gamma^2 > 1$.

APPENDIX C: ESTIMATION OF THE EIGENVALUES OF THE STABILITY MATRIX

Here we will estimate the eigenvalues of I_0,

$$I_{0;j,l} = (d-1)\delta_{j,l} + \frac{1}{2\sigma^2}\frac{1}{(2\pi)^{d-1}}\int \frac{\theta(\kappa^2-\gamma^2)}{\kappa^2}\, e^{ik\cdot(e_l+e_j)}\, d^{d-1}k$$

Because of the step function the wave vector k_j in the direction perpendicular to the z axis is restricted to $k_j \sim 0$ or $k_j \sim \pm\pi$ if $|\gamma| \sim 1$. Therefore, the diagonal elements of I_0 are

$$I_{0;j,j} = (d-1) + \frac{1}{2\sigma^2}\frac{1}{(2\pi)^{d-1}}\int \frac{\theta(\kappa^2-\gamma^2)}{\kappa^2}\, e^{2ik_j}\, d^{d-1}k \sim d-1\,\frac{1}{2\gamma^2} \tag{C.1}$$

since

$$\sigma^2 \sim \frac{1}{(2\pi)^{d-1}}\int \theta(\kappa^2-\gamma^2)\, d^{d-1}k \tag{C.2}$$

The off-diagonal elements (i.e., $j \neq l$) of I_0 can be estimated as

$$|I_{0;j,l}| = \left| \frac{1}{2\sigma^2} \frac{1}{(2\pi)^{d-1}} \int \frac{\theta(\kappa^2 - \gamma^2)}{\kappa^2} e^{ik \cdot (e_l + e_j)} d^{d-1}k \right| \leqslant \frac{1}{2\gamma^2} \tag{C.3}$$

Suppose I_0' is the stability matrix consisting only of the off-diagonal elements of I_0. Its eigenvalues λ_j' can be estimated from above as

$$|\lambda_j'|^2 \leqslant \mathrm{Tr}(I_0'^2) = \sum_{j,l=0}^{2(d-1)} (I_0')_{j,l} (I_0')_{l,j} \leqslant \frac{1}{4\gamma^4} 2(d-1)[2(d-1)-1] \tag{C.4}$$

This implies for the eigenvalues of I_0

$$\begin{aligned} \lambda_j &\sim d - 1 + \frac{1}{2\gamma^2} + \lambda_j' \\ &\geqslant d - 1 + \frac{1}{2\gamma^2} - |\lambda_j'| \\ &\geqslant d - 1 + \frac{1}{\gamma^2}\left[\frac{1}{2} - (d-1)\left(\frac{2d-3}{2d-2}\right)^{1/2}\right] \\ &> 0 \end{aligned} \tag{C.5}$$

for $|\gamma| \sim 1$.

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